Voltage-Current curves for small Josephson junction arrays

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We compute the current voltage characteristic of a chain of identical Josephson circuits characterized by a large ratio of Josephson to charging energy that are envisioned as the implementation of topologically protected qubits. We show that in the limit of small coupling to the environment it exhibits a non-monotonous behavior with a maximum voltage followed by a parametrically large region where $V \propto 1/I$. We argue that its experimental measurement provides a direct probe of the amplitude of the quantum transitions in constituting Josephson circuits and thus allows their full characterization.

I. INTRODUCTION

In the past years, the dramatic experimental progress in the design and fabrication of quantum two level systems in various superconducting circuits[1] has raised a hope that such solid state devices could eventually serve as basic logical units in a quantum computer (qubits). However, a very serious obstacle on this path is the ubiquitous decoherence, which in practice limits the typical life-time of quantum superpositions of two distinct logical states of a qubit to microseconds. This is far from being sufficient to satisfy the requirements for implementing quantum algorithms and providing systematic error correction.[2]

This has motivated us to propose some alternative ways to design Solid-State qubits, that would be much less sensitive to decoherence than those presently available. These protected qubits are finite size Josephson junction arrays in which interactions induce a degenerate ground-state space characterized by the remarkable property that all the local operators induced by couplings to the environment act in the same way as the identity operator. These models fall in two classes. The first class is directly inspired by Kitaev's program of topological quantum computation,[3] and amounts to simulating lattice gauge theories with small finite gauge groups by a large Josephson junction lattice.[4,5,6] The second class is composed of smaller arrays with sufficiently large and non-Abelian symmetry groups allowing for a persistent ground-state degeneracy even in the presence of a noisy environment.[7,8] All these systems share the property that in the classical limit for the local superconducting phase variables (i.e. when the Josephson coupling is much larger than the charging energy), the ground-state is highly degenerate. The residual quantum processes within this low energy subspace lift the classical degeneracy in favor of macroscopic coherent superpositions of classical ground-states. The simplest example of such system is based on chains of rhombi (Fig. 1) frustrated by magnetic field flux $\Phi = \Phi_0/2$ that ensures that in the classical limit each rhombus has two degenerate states.[8]

Practically, it is important to be able to test these ar-

rays and optimize their parameters in relatively simple experiments. In particular one needs the means to verify the degeneracy of the classical ground states, the presence of the quantum coherent processes between them and measure their amplitude. Another important parameter is the effective superconducting stiffness of the fluctuating rhombi chain. The classical degeneracy and chain stiffness can be probed by the experiments discussed in[9]; they are currently being performed[10]. The idea is that a chain of rhombi threaded individually by half a superconducting flux quantum, the non-dissipative current is carried by charge $4e$ objects,[11,12] so that the basic flux quantum for a large closed chain of rhombi becomes $h/(4e)$ instead of $h/(2e)$ which can be directly observed by measuring the critical current of the loop made from such chain and a large Josephson junction.

The main goal of the present paper is to discuss a practical way to probe directly the quantum coherence associated with these tunneling processes between macroscopically distinct classical ground-states. In principle, it is relatively simple to implement, since it amounts to measuring the average dc voltage generated across a finite Josephson junction array in the presence of a small current bias (i. e. this bias current has to be smaller than the critical current of the global system). The physical mechanism leading to this small dissipation is very interesting by itself; it was orinally discussed in a seminal paper by Likharev and Zorin[16] in the context of a single Josephson junction. Consider one element (single junction or a rhombus) of the chain, and denote by ϕ the phase difference across this element. When it is disconnected from the outside world, its wave-function Ψ is $2\pi\zeta$-periodic in ϕ where $\zeta = 1$ for a single junction and $\zeta = 1/2$ for a rhombus. This reflects the quantization of the charge on the island between the elements which can change by integer multiples of $2e/\zeta$. If ϕ is totally classical, the element's energy is not sensitive to the choice of a quasi-periodic boundary condition of the form $\Psi(\phi + 2\pi\zeta) = \exp(i2\pi\zeta q)\Psi(\phi)$, where q represents the charge difference induced across the rhombus. In the presence of coherent quantum tunneling processes for ϕ, the energy of the element $\epsilon(q)$ will acquire q-dependence, with a bandwidth directly related to the basic tunnel-

ing amplitude. Whereas q is constrained to be integer for an isolated system, it is promoted to a genuine continuous degree of freedom when the array is coupled to leads and therefore to a macroscopic dissipative environment. So, as emphasized by Likharev and Zorin[16], the situation becomes perfectly analogous to the Bloch theory of a quantum particle in a one-dimensional periodic potential, where the phase ϕ plays the role of the position, and q of the Bloch momentum. A finite bias current tilts the periodic potential for the phase variable, so that in the absence of dissipation, the dynamics of the phase exhibits Bloch oscillations, very similar to those which have been predicted[17] and observed[18,19] for electrons in semi-conductor super-lattices. If the driving current is not too large, it is legitimate to neglect inter-band transitions induced by the driving field, and one obtains the usual spectrum of equally spaced localized levels often called a Wannier-Stark ladder. In the presence of dissipation, these Wannier-Stark levels acquire a finite life-time, and therefore the time-evolution of the phase variable is characterized by a slow and uniform drift superimposed on the faster Bloch oscillations. This drift is translated into a finite dc voltage by the Josephson relation $2eV = \hbar(d\phi/dt)$. This voltage decreases with current until one reaches the current bias high enough to induce the interband transition. At this point the phase starts to slide down fast and the junction switches into a normal state. In the context of Josephson junctions these effects were first observed in the experiments on Josephson contacts with large charging energy[20,21,22,23] and more recently[24,25] in the semiclassical (phase) regime of interest to us here. Bloch oscillations in the quantronium circuit driven by a time-dependent gate voltage have also been recently observed.[26]

This picture holds as long as the dissipation affecting the phase dynamics is not too strong, so that the radiative width of the Wannier-Stark levels is smaller than the nearest-level spacing (corresponding to phase translation by $2\pi\zeta$) that is proportional to the bias current. This provides a lower bound for the bias current which has to be compatible with the upper bound coming from the condition of no inter-band transitions. As we shall see, this requires a large real part of the external impedance $Z_\omega \gg R_Q$ as seen by the element at the frequency of the Bloch oscillation, where the quantum resistance scale is $R_Q = h/(4e^2)$. This condition is the most stringent in order to access experimentally the phenomenon described here. Note that this physical requirement is not limited to this particular experimental situation, because any circuit exploiting the quantum coherence of phase variables, for instance for quantum information processing, has to be imbedded in an environment with a very large impedance in order to limit the additional quantum fluctuations of the phase induced by the bath. The intrinsic dissipation of Josephson elements will of course add to the dissipation produced by external circuitry, but we expect that in the quantum regime (i.e. with sizable phase fluctuations) considered here, this additional impedance

will be of order of R_Q at the superconducting transition temperature, and will grow exponentially below. Thus, the success of the proposed measurements is also a test of the quality of the environment for the circuits intended to serve as protected qubits.

In many physical realizations Z_ω has a significant frequency dependence and the condition $Z_\omega \gg R_Q$ is satisfied only in a finite frequency range $\omega_{\max} > \omega > \omega_{\min}$. This situation is realized, for example, when the Josephson element is decoupled from the environment by a long chain of larger Josephson junctions (Section V). In this case the superconducting phase fluctuations are suppressed at low frequencies implying that a phase coherence and thus Josephson current reappears at these scales. The magnitude of the critical current is however strongly suppressed by the fluctuations at high frequencies. This behavior is reminiscent of the reappearance of the Josephson coupling induced by the dissipative environment observed in[27]. At higher energy scales fluctuations become relevant, the phase exhibits Bloch oscillations resulting in the insulating behavior described above. Thus, in this setup one expects a large hierarchy of scales: at very low currents one observes a very small Josephson current, at larger currents an almost insulating behavior and finally a switching into the normal state at largest currents.

In the case of a chain of identical elements, the total dc voltage is additive, but Bloch oscillation of different elements might happen either in phase or in antiphase. In the former case the ac voltages add increasing the dissipation in the external circuitry; while in the latter case the dissipation is low and the individual elements get more decoupled from the environment. As we show in Section III a small intrinsic dissipation of the individual elements is crucial to ensure the antiphase scenario.

This paper is organized as follows. In section II, we present a semi-classical treatment of the voltage versus bias current curves for a single Josephson element. We show that this gives an accurate way to measure the effective dispersion relation $\epsilon(q)$ of this element, which fully characterizes its quantum transition amplitude. Further, we show that application of the ac voltage provides a direct probe of the periodicity (2π versus π) of each element. In Section III we consider the chain of these elements and show that under realistic assumptions about the dynamics of individual elements, it provides much more efficient decoupling from the environment. Section IV focusses on the dispersion relation expected in a practically important case of a fully frustrated rhombus which is the building block for the protected arrays considered before.[5,8] In this case, the band structure has been determined by numerical diagonalizations of the quantum Hamiltonian. An important result of this analysis is that even in the presence of relatively large quantum fluctuations, the effective band structure is always well approximated by a simple cosine expression. Finally, in section V we discuss the conditions for the experimental implementation of this measurement procedure and

the full $V(I)$ characteristics expected in realistic setup. After a Conclusion section, an Appendix presents a full quantum mechanical derivation of the dc voltage when the bias current is small enough so that inter-band transitions can be neglected, and large enough so that the level decay rate can simply be estimated from Fermi's golden rule.

II. SEMI-CLASSICAL EQUATIONS FOR A SINGLE JOSEPHSON ELEMENT

Let us consider the system depicted on Fig. (1). In the absence of the current source, the energy of the one dimensional chain of N Josephson elements is a $2\pi\zeta$ periodic function of the phase difference ϕ across the chain. The current source is destroying this periodicity by introducing the additional term $-\hbar(I/2e)\phi$ in the system's Hamiltonian. Because ϕ is equal to the sum of phase differences across all the individual elements, it seems that the voltage generated by the chain is N times the voltage of a chain reduced to a single element. This is, however, not the case: the individual elements are coupled by the common load, and furthermore, as we show in the next section, their collective behavior is sensitive to the details of the single element dynamics. In this section, we consider the case of a single Josephson element ($N = 1$), rederive the results of Likharev and Zorin[16] for single Josephson contact and generalize them for more complicated structures such as rhombus and give analytic equations convenient for data comparison.

The dynamics of a single Josephson contact is analogous to the motion of a quantum particle (with a charge e) in a one-dimensional periodic potential (with period a) in the presence of a static and uniform force F, the phase-difference ϕ playing the role of the spatial coordinate x of the particle.[16] In the limit of a weak external force, it is natural to start by computing the band structure $\epsilon_n(k)$ for k in the first Brillouin zone $[-\pi/a, \pi/a]$, n being the band label. A first natural approximation is to neglect interband transitions induced by the driving field. This is possible provided the Wannier-Stark energy gap $\Delta_\mathrm{B} = Fa$ is smaller than the typical band gap Δ in zero external field. As long as Δ_B is also smaller than the typical bandwidth W, the stationary states of the Schrödinger equation spread over many (roughly W/Δ_B) periods, so we may ignore the discretization (i.e. one quantum state per energy band per spacial period) imposed by the projection onto a given band. We may therefore construct wave-packets whose spacial extension Δx satisfies $a \ll \Delta x \ll aW/\Delta_\mathrm{B}$, and the center of such a wave-packet evolves according to the semi-classical equations:

$$\frac{dx}{dt} = \frac{1}{\hbar}\frac{d\epsilon_n(k)}{dk} \tag{1}$$

$$\frac{dk}{dt} = \frac{1}{\hbar}F \tag{2}$$

In the presence of dissipation, the second equation is modified according to:

$$\frac{dk}{dt} = \frac{1}{\hbar}F - \frac{m^*}{\hbar\tau}\frac{dx}{dt} \tag{3}$$

where m^* is the effective mass of the particle in the n-th band and τ is the momentum relaxation time introduced by the dissipation.
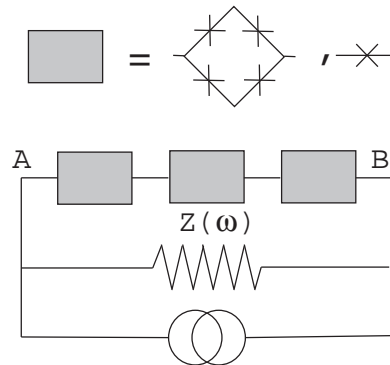

FIG. 1: The experimental setup discussed in this paper: a chain of identical building blocks represented by shaded rectangle that are biased by the external current source characterized by the impedance $Z(\omega)$. The internal structure of the block that is considered in more detail in the following sections is either a rhombus (4 junction loop) frustrated by half flux quantum, or a single Josephson junction but the the results of the section II can be applied to any circuit of this form provided that the junctions in the elementary building blocks are in the phase regime, i.e. $E_J \gg E_C$.

In the context of a Josephson circuit, we have to diagonalize the Hamiltonian describing the array as a function of the pseudo-charge q associated with the $2\pi\zeta$ periodic phase variable ϕ. The quantity q controls the periodic boundary condition imposed on ϕ, namely the system's wave-function is multiplied by $\exp(i2\pi q)$ when ϕ is increased by $2\pi\zeta$. From this phase-factor, we see that the corresponding Brillouin zone for q is the interval $[-1/2, 1/2]$. For a simple Josephson contact ($\zeta = 1$), the fixed value of q means that the total number of Cooper pairs on the site carrying the phase ϕ is equal to q plus an arbitrary integer. For a doubly periodic element, such as rhombus ($\zeta = 1/2$), charge is counted in the units of $4e$. To simplify the notations we assume usual 2π periodicity ($\zeta = 1$) in this and the following Sections and restore the ζ-factors in Sections IV, V. From the band structure $\epsilon_n(q)$, we may write the semi-classical equations of motion in the presence of the bias current I and the outer impedance Z as:

$$\frac{d\phi}{dt} = \frac{1}{\hbar}\frac{d\epsilon_n(q)}{dq} \tag{4}$$

$$\frac{dq}{dt} = \frac{I}{2e} - \frac{Z_\mathrm{Q}}{Z}\frac{d\phi}{dt} \tag{5}$$

where we used the Josephson relation for the voltage drop V across the Josephson element as $V = (\hbar/2e)(d\phi/dt)$ and defined $Z_\mathrm{Q} = \hbar/(4e^2)$.

This semi-classical model exhibits two different regimes. Let us denote by $\omega_{\max}$ the maximum value of the "group velocity" $|d\epsilon_n(q)/(\hbar dq)|$. If the driving current is small ($I < I_c = 2e\omega_{\max}Z_Q/Z$), it is easy to see that after a short transient, the system reaches a stationary state where q is constant and:

$$\frac{d\phi}{dt} = \frac{I}{2e}\frac{Z}{Z_Q} \qquad (6)$$

that is: $V = ZI$. Thus, at $I < I_c$ the current flows entirely through the external impedance, i.e. the Josephson elements become effectively insulating due to quantum phase fluctuations. Indeed, a Bloch state written in the phase reprentation corresponds to a fixed value of the pseudo-charge q and non-zero dc voltage $(1/2e)(d\epsilon_n/dq)$. Note that the measurement of the maximal value V_c of the voltage on this linear branch directly probes the spectrum of an individual Josephson block, because $V_c = \hbar\omega_{\max}/2e$

At stronger driving ($I > I_c$), it is no longer possible to find a stationary solution for q. The system enters therefore a regime of Bloch oscillations. In the absence of dissipation ($Z/Z_Q \to \infty$), the motion is periodic in time for both ϕ and q. A small but finite dissipation preserves the periodicity in q, but induces an average drift in ϕ or equivalently a finite dc voltage. To see this, we first note that the above equations of motion imply:

$$\frac{dq}{dt} = \frac{I}{2e} - \frac{Z_Q}{Z}\frac{1}{\hbar}\frac{d\epsilon_n}{dq} \qquad (7)$$

Since the right-hand side is a periodic function of q with period 1, $q(t)$ is periodic with the period $T(I)$ given by:

$$T(I) = \int_{-1/2}^{1/2} f(q)dq \qquad (8)$$

with

$$f(q) = \left(\frac{I}{2e} - \frac{Z_Q}{Z}\frac{1}{\hbar}\frac{d\epsilon_n}{dq}\right)^{-1} \qquad (9)$$

On the other hand, the instantaneous dissipated power reads:

$$\frac{d}{dt}\left(\epsilon_n(q) - \frac{\hbar I}{2e}\phi\right) = -\hbar\frac{Z_Q}{Z}(\frac{d\phi}{dt})^2 \qquad (10)$$

Because $q(t)$ is periodic, averaging this expression over one period gives:

$$\langle\frac{d\phi}{dt}\rangle = \frac{2e}{I}\frac{Z_Q}{Z}\langle(\frac{d\phi}{dt})^2\rangle \qquad (11)$$

or, equivalently:

$$\langle V\rangle = \frac{\hbar}{I}\mathrm{Re}\left(\frac{Z_Q}{Z}\right)\langle(\frac{d\phi}{dt})^2\rangle \qquad (12)$$

Using the equations of motion, we get more explicitly:

$$\langle V\rangle = \frac{1}{4e^2 I}\mathrm{Re}\left(\frac{Z_Q}{Z_\omega}\right)\frac{\int_{-1/2}^{1/2}(\frac{d\epsilon_n}{dq})^2 f(q)dq}{\int_{-1/2}^{1/2} f(q)dq} \qquad (13)$$

Here we emphasized by the subscript that Z_ω might have some frequency dependence. As we show in Appendix, the dissipation actually occurs at the frequency of Bloch oscillations that becomes $\omega_B = 2\pi I/2e$ in the limit of large currents. In the limit of large currents, $I \gg I_c$, (that can be achieved for large impedances) we may approximate $f(q)$ by a constant, so the voltage is given by the simpler expression:

$$\langle V(I \gg I_c)\rangle = \frac{1}{4e^2 Z_\omega I}\int_{-1/2}^{1/2}(\frac{d\epsilon_n}{dq})^2 dq \qquad (14)$$

On the other hand, when I approaches I_c from above, Bloch oscillations become very slow and $f(q)$ is strongly peaked in the vicinity of the maximum of the group velocity. Since this velocity is in general a smooth function of q, we get in this limit for the maximal dc voltage:

$$V_c = \frac{\hbar^2\omega_{\max}^2}{4e^2 Z_0 I_c} = Z_0 I_c \qquad (15)$$
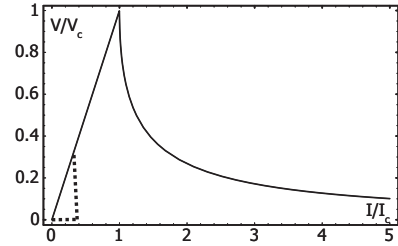

FIG. 2: Typical $I - V$ curve of a single Josephson element measured by a circuits shown in Fig. 1

In the simplest case of a purely harmonic dispersion, $\epsilon(q) = 2w\cos 2\pi q$, the maximal voltage $V_c = 4\pi w/(2e)$. If one can further neglect the frequency dependency of Z, the $V(I)$ can be computed analytically:

$$\langle V\rangle = ZI \quad I < I_c \qquad (16)$$

$$\langle V\rangle = ZI_c\frac{I_c}{I + \sqrt{I^2 - I_c^2}} \quad I > I_c \qquad (17)$$

We show this dependence in Fig. 2. This expression (16), (17) is related to the known result for $Z \ll Z_Q$[13,14] by the duality[15] transformation:

$$\begin{aligned} V &\to I, \\ I &\to V, \\ Z &\to \frac{1}{Z}. \end{aligned}$$

The semi-classical approximation is valid when the oscillation amplitude of the superconducting phase is much

larger than 2π, which allows the formation of the semi-classical wave-packets. When I is much larger than I_c, this oscillation amplitude is equal to $2eW/\hbar I$, where W is the total band-width of $\epsilon_n(q)$. This condition also ensures that the work done by the current source when the phase increases by 2π is much smaller than the band-width. In order to observe the region of negative differential resistance, corresponding to the regime of Bloch oscillations, we require therefore that:

$$\frac{2\pi\hbar I_c}{2e} \ll W \simeq \frac{2eV_c}{\pi}, \qquad (18)$$

where the last equality becomes exact in the case of a purely harmonic dispersion. This translates into:

$$Z \gg R_Q. \qquad (19)$$

For large currents one can compute dc voltage directly by using the golden rule (without semiclassics); we present the results in Appendix A. The result is consistent with the large I limit of Eq. (17), $\langle V \rangle = V_c^2/2ZI$. Deep in the classical regime ($E_J \gg E_C$), the bandwidth and the generated voltage become exponentially small. In this regime the bandwidth is much smaller than the energy gaps, so these formulas are applicable (assuming (19) is satisfied) until the splitting between Wannier-Stark levels becomes equal to the first energy gap given by the Josephson plasma frequency, i.e. for $I < e\omega_J/\pi$. Upon a further increase of the driving current in this regime the generating voltage experiences resonant increase for each splitting that is equal to the energy gap: $I_k = e(E_k - E_0)/\pi$. Physically, at these currents the phase slips are rare events that lead to the excitation of the higher levels at a new phase value that are followed by their fast relaxation. At very large energies, the bandwidth of these levels becomes larger than their decay rate due to relaxation, $(R_Q/Z)E_C$. At these driving currents, the system starts to generate large voltage and switches to a normal state. At a very large E_J this happens at the driving currents very close to the Josephson critical current $2eE_J$, but in a numerically wide regime of $100 \gtrsim E_J/E_C \gtrsim 10$ the generated voltage at low curents is exponentially small but switching to the normal state occurs at significantly smaller currents than $2eE_J$.

In the intermediate regime where E_J and E_C are comparable, we expect a band-width comparable to energy gaps so that the range of application of the quantum derivation is not much larger than the one for the semi-classical approach.

Negative differential resistance associated to Bloch oscillations has been predicted long ago,[28] and observed experimentally[29] in the context of semi-conductor superlattices. For Josephson junctions in the cross-over regime ($E_J/E_C \simeq 1$), a negative differential resistance has been observed in a very high impedance environment,[24] in good agreement with earlier theoretical predictions.[30] More recently, the $I-V$ curve of the type shown on Fig 2 have been reported on a junction with a ratio $E_J/E_C = 4.5$[25]. These experiments show good

agreement with a calculation which takes into account the noise due to residual thermal fluctuations in the resistor.[31]

Although the above results allows the extraction of the band structure of an individual Josephson block from the measurement of dc $I-V$ curves, the interpretation of actual data may be complicated by frequency dependence of the external impedance Z_ω. Additional information independent on Z_ω can be obtained from measuring the dc $V(I)$ characteristics in the circuit driven by an additional ac current. In this situation, the semi-classical equations of motion become:

$$\frac{d\phi}{dt} = \frac{1}{\hbar}\frac{d\epsilon_n(q)}{dq} \qquad (20)$$

$$\frac{dq}{dt} = \frac{I + I'\cos(\omega t)}{2e} - \frac{Z_Q}{Z}\frac{d\phi}{dt} \qquad (21)$$

A small ac driving amplitude I' strongly affects the $V(I)$ curve only in the vicinity of resonances where $n\omega_B(I_R) = m\omega$, with m and n integers. The largest deviation occurs for $m = n = 1$. Furthermore, for $I' \ll I$ the terms with $m > 1$ are parametrically small in I'/I while for $I \gg I_c$ the terms with $n > 1$ are parametrically small in I_c/I. Experimental determination of the resonance current, I_R, would allow a direct measurement of the Bloch oscillation frequency and thus the periodicity of the phase potential (see next Section). Observation of these mode locking properties have in fact provided the first experimental evidence of Bloch oscillations in a single Josephson junction.[20,21]

We now calculate the shape of $V(I)$ curve in the vicinity of $m = n = 1$ point when both $I' \ll I$ and $I \gg I_c$. We denote by $\phi_0(t)$ and $q_0(t)$ the time-dependent solutions of the equations at $I = I_R$ in the absence of ac driving current. We shall look for solutions which remain close to $\phi_0(t)$ and $q_0(t)$ at all times and expand them in small deviations $\phi_1 = \phi - \phi_0$, $q_1 = q - q_0$. We can always assume that q_1 has no Fourier component at zero frequency because such component can be eliminated by a time translation applied to q_0. The equations for ϕ_1, q_1 become

$$\frac{d\phi_1}{dt} = \frac{1}{\hbar}\epsilon_n''(q_0)q_1 \qquad (22)$$

$$\frac{dq_1}{dt} = \frac{I - I_R + I'\cos(\omega t)}{2e} - \frac{Z_Q}{Z}\frac{d\phi_1}{dt} \qquad (23)$$

Because the main component of $\frac{d^2\epsilon_n(q_0)}{dq^2}$ oscillates with frequency ω and q_1 has no dc component, the average value of the voltage $\frac{d\phi_1}{dt}$ is due to the part of q_1 that oscillates with the same frequency, $q_{1\omega} = I'/(2e\omega)\sin(\omega t)$. Because $q_0 = \omega(t - t_0) + \chi(\omega(t - t_0))$ where $\chi(t)$ is a small periodic function, the first equation implies that

$$< \frac{d\phi_1}{dt} > = \langle\frac{1}{\hbar}\epsilon_n''(\omega(t - t_0))\sin(\omega t)\rangle$$

$$< \frac{1}{\hbar}\int_0^1 \epsilon_n''(q)\cos(2\pi q)dq$$

The deviation q_1 remains small only if the constant parts cancel each other in the right hand side of the equation (23). This implies

$$< \frac{d\phi_1}{dt} >= \frac{Z}{Z_Q} \frac{I - I_R}{2e} < \frac{1}{\hbar} \int_0^1 \epsilon_n(q) \cos(2\pi q) dq \quad (24)$$

We conclude that in the near vicinity of the resonances the increase of the current does not lead to additional current through the Josephson circuit, so the relation between current and voltage becomes linear again $\delta V = Z \delta I$. In other words, the Josephson circuit becomes insulating with respect to current increments. The width of this region (in voltage) is directly related to the first moment of the energy spectrum of the Josephson block providing one with the direct experimental probe of this quantity. In particular, a Josephson element such as rhombus in a magnetic flux somewhat different from $\Phi_0/2$ displays a phase periodicity 2π but a very strong deviations from a simple $\cos 2\pi q$ spectrum that will manifest themselves in first moment of the spectrum. Note finally, that the discussion above assumes that the external impedance Z_ω has no resonances in the important frequency range. The presence of such resonances will modify significantly the observed $V(I)$ curves because it would provide an efficient mechanism for the dissipation of Bloch (or Josephson) oscillations at this frequency.

III. CHAIN OF JOSEPHSON ELEMENTS

We shall first consider the simplest example of a two-element chain, because it captures the essential physics. This chain is characterized by two phase differences (ϕ_1 and ϕ_2) and two pseudo-charges (q_1 and q_2). The equations of motion for the pseudo-charges (5) implies that the charge difference $q_1 - q_2$ is constant, because the currents flowing through these elements are equal, and thus the right-hand sides of the evolution equations (5) are identical. Because of this conservation law, even the long-term physical properties depend on the initial conditions. Similar problems have already been discussed in the context of a chain of Josephson junctions driven by a current larger than the critical current.[32,33,34,35] This unphysical behavior disappears if we take into account the dissipation associated with individual elements. Physically, it might be due to stray charges, two-level systems, quasiparticles, phonon emission, etc.[36,37] A convenient model for this dissipation is to consider an additional resistor in parallel with each junction. For the sake of simplicity, we assume that each element has a low energy band with a simple cosine form. This physics is summarized by the equations:

$$\dot{\phi}_j = 4\pi w \sin 2\pi q_j \quad (25)$$

$$\dot{q}_j = \frac{1}{2e} \left(I - \frac{1}{2eZ} \sum_i \dot{\phi}_i - \frac{1}{2eR_j} \dot{\phi}_j \right) \quad (26)$$

Eliminating the phases gives:

$$(\dot{q}_1 + \Omega_1 \sin 2\pi q_1) = \nu - \frac{\nu_0}{2}(\sin 2\pi q_1 + \sin 2\pi q_2)$$

$$(\dot{q}_2 + \Omega_2 \sin 2\pi q_2) = \nu - \frac{\nu_0}{2}(\sin 2\pi q_1 + \sin 2\pi q_2)$$

where

$$\Omega_i = \frac{4\pi w}{(2e)^2 R_i},$$

$$\nu = \frac{I}{2e}$$

$$\nu_0 = \frac{8\pi w}{(2e)^2 Z}$$

Here we allowed for different effective resistances associated with each element because this has an important effect on their dynamics. Indeed the difference between the currents flowing through the resistors changes the charge accumulated at the middle island and therefore violates the conservation law mentioned before. Using the notations $\delta\Omega = (\Omega_2 - \Omega_1)/2$ and $q_\pm = (q_2 \pm q_1)/2$, we have:

$$\dot{q}_- + \Omega \sin 2\pi q_- \cos 2\pi q_+ +$$
$$+ \delta\Omega \cos 2\pi q_- \sin 2\pi q_+ = 0 \quad (27)$$
$$\dot{q}_+ + (\nu_0 + \Omega) \sin 2\pi q_+ \cos 2\pi q_- -$$
$$- \delta\Omega \sin 2\pi q_- \cos 2\pi q_+ = \nu \quad (28)$$

Significant quantum fluctuations imply that internal resistance of the element $R \sim Z_Q$ for individual elements at $T \lesssim T_C$; at lower temperature it grows exponentially. Thus, in a realistic case $R \gg Z$ which implies that $\Omega_i \ll \nu$. In the insulating regime the equations (27-28) have stable stationary solution $(\nu_0 + \Omega) \sin 2\pi q_+ = \nu$, $q_- = 0$. This solution exists for $(\nu_0 + \Omega) < \nu$, i.e. if the voltage drop across both junctions does not exceed $V_c = 8\pi w/(2e)$. The conducting regime occurs when $\nu > (\nu_0 + \Omega)$; to simplify the analytic calculations we assume that $\nu \gg \nu_0$. This allows to solve the equations (27-28) by iterations in all non-linear terms. In the absence of non-linearity $q_+ = \nu t$, $q_- = const$; the first iteration gives periodic corrections $\propto \cos 2\pi\nu t$. Averaging the result of the second order iteration over the period we get

$$\langle \dot{q}_- \rangle = -\frac{\delta\Omega}{2\nu} \left[\nu_0 \cos^2 2\pi q_- + 2\Omega \right] \quad (29)$$

The second term in the right hand side of this equation is much smaller than the first if $\Omega \ll \nu_0$. In its absence the dynamics of q_- has fixed points at $\cos 2\pi q_- = 0$. At these fixed points the periodic potentials generated by individual elements cancel each other and the dissipation in external circuitry (which is proportional to $\cos^2(2\pi q_-)$) is strictly zero. In a general case the equation (29) has solution

$$\cos^2(2\pi q_-) = \frac{1}{1 + \frac{\nu_0 + 2\Omega}{2\Omega} \tan^2 \left[\frac{\pi}{\nu} \delta\Omega \sqrt{2\Omega(\nu_0 + 2\Omega)} t \right]}$$

that corresponds to the short bursts of dissipation in external circuitry that occur with low frequency $\nu_b = \frac{2}{\nu}\delta\Omega\sqrt{2\Omega(\nu_0 + 2\Omega)}$. The average value of $\cos^2(2\pi q_-)$

$$< \cos^2(2\pi q_-) > = \frac{1}{1 + \sqrt{\frac{\nu_0 + 2\Omega}{2\Omega}}} \approx \sqrt{\frac{2\Omega}{\nu_0 + 2\Omega}}$$

is small implying that the effective dissipation introduced by the external circuitry is strongly suppressed because the pseudocharge oscillations on different elements almost cancel each other. The effective impedance of the load seen by individual junction is strongly increased:

$$Z_{eff} = \sqrt{\frac{\nu_0 + 2\Omega}{2\Omega}} Z \qquad (30)$$

Similar to a single element case discussed in the previous Section, an additional dissipation in the external circuit implies dc current across the Josephson chain

$$V = V_c \frac{I_c}{2I} \; I \gg I_c = V_c/Z_{eff}$$

We conclude that a chain of Josephson elements has a current-voltage characteristics similar to the one of the single element with one important difference: the effective impedance of the external circuitry is strongly enhanced by the antiphase locking of the individual Josephson elements. In particular, it means that the condition $Z \gg R_Q$ is much easier to satisfy for the chain of the elements than for a single element. The analytical equations derived here describe the chain of two elements but it seems likely that similar suppression of the dissipation should occur in longer chains.

To substantiate this claim, lets us generalize the averaging method which led to Eq. (29) for $N = 2$. The coupled equations of motion read:

$$\dot{q}_j + \Omega_j \sin 2\pi q_j = \nu - \frac{\nu_0}{2}\sum_{k=1}^{N} \sin 2\pi q_k \qquad (31)$$

To second order in Ω_j and ν_0, the averaged equations of motion are:

$$\begin{aligned} \langle \dot{q}_j \rangle = & -\frac{\Omega_j^2}{2\nu} - \frac{\nu_0}{4\nu}\sum_{k=1}^{N}\Omega_k \\ & -\frac{\nu_0^2}{8\nu}\sum_{k,l}\cos(2\pi(q_k - q_l)) \\ & -\frac{\nu_0 \Omega_j}{4\nu}\sum_{k=1}^{N}\cos(2\pi(q_j - q_k)) \end{aligned} \qquad (32)$$

This set of coupled equations is similar to the Kuramoto model for coupled rotors[38] defined as:

$$\dot{q}_j = \omega_j - \frac{K}{N}\sum_{k=1}^{N}\sin(2\pi(q_j - q_k) + \alpha) \qquad (33)$$

The equation of motion (33) exhibits synchronisation of a finite fraction of the rotors only for $K > K_c(\alpha)$.[39,40] The last term in Eq. (32) is equivalent to the interaction term of Kuramoto model with $\alpha = \pi/2$. The additional (third) term in the model (32) is the same for all oscillators, it is thus not correlated with individual q_j and thus can not directly lead to their synchronisation. Remarkably, it turns out that for model (33) $K_c(\alpha = \pi/2) = 0$[39,40], suggesting that in our case, synchronization never occurs on a macroscopic scale. Note that the coupling K arising from Eq. (32) is not only j-dependent, but it is also proportional to N. This could present a problem in the infinite N limit, but should not present a problem in a finite system. It is striking to see that $\alpha = \pi/2$ is the value for which synchronization is the most difficult.

IV. ENERGY BANDS FOR A FULLY FRUSTRATED JOSEPHSON RHOMBUS

In order to apply general results of the previous section to the physical chains made of Josephson junctions or more complicated Josephson circuits we need to compute the spectrum of these systems as a function of the pseudocharge q conjugated to the phase across these elements. In all cases the superconducting phase in Josephson devices fluctuates weakly near some classical value ϕ_0 where the Josephson energy has a minimum in the limit $E_J/E_C \gg 1$. In the vicinity of the minimum, the phase Hamiltonian is $H = -4E_C\frac{d^2}{d\phi^2} + \frac{1}{2}E''(\phi_0)(\phi - \phi_0)^2$, so a higher energy state of the individual element (at a fixed q) can be approximated by one of the oscillator $E_n = (n + \frac{1}{2})\omega_J$ where the Josephson plasma frequency $\omega_J = \sqrt{8E''(\phi_0)E_C} \approx \sqrt{8E_J E_C}$. The Josephson energy is periodic in the phase with the period 2π but the amplitude of the transitions between these minima is exponentially small:

$$w = a\hbar\omega_J(E_J/E_C)^{1/4}\exp(-c\sqrt{E_J/E_C})$$

where $a, c \sim 1$. In this limit one can neglect the contribution of the excited states (separated by a large gap ω_J) to the lower band, so the low energy spectrum acquires a simple form $\epsilon(q) = 2w\cos 2\pi q$. The numerical coefficients c, a in the formulae for the transition amplitude depend on the element construction. For a single junction $a_s = 8\,2^{1/4}/\sqrt{\pi}$, $c_s = \sqrt{8}$ while for the rhombus $a_r \approx 4.0$, $c_r \approx 1.61$. In case of the rhombus in magnetic field with flux $\Phi_0/2$ the Hamiltonian is periodic in phase with period π provided that the rhombus is symmetric along its horizontal axis: indeed in this case the combination of the time reversal symmetry and reflection ensures that the Josephson energy has a minimum for $\phi_0 = \pm\pi/2$. Thus, in this case the period in q doubles and the low energy band becomes $\epsilon(q) = 2w\cos\pi q$. The maximal voltage generated by the chain of N such elements at $I = I_c = (8\pi\zeta ew/\hbar)(Z_Q/Z)$ is

$$V_c = N\frac{4\pi\zeta w}{2e} \tag{34}$$

The voltage generated at larger currents depend on the collective behaviour of the elements in the chain. For a single element it is simply

$$\langle V(I)\rangle = \frac{(2\pi\zeta w)^2}{e^2 Z_\omega}\frac{1}{I + \sqrt{I^2 - I_c^2}},$$

For more than one element the total volage is sufficiently reduced due to the antiphase correlations. Generally, one expects that

$$\langle V(I)\rangle = N\frac{(2\pi\zeta w)^2}{e^2 Z_\omega^{eff}}\frac{1}{I + \sqrt{I^2 - I_c^2}}, \tag{35}$$

where Z_ω^{eff} is the effective impedance of the environment affecting each Josephson element which is generally much larger than its 'bare' impedance Z_ω. For two elements the exact solution (see previous Section) gives $Z_\omega^{eff} \approx \sqrt{RZ}$ that shows the increase of the effective impedance by a large factor $\sqrt{R/Z}$. We expect that a similar enhancement factor appears for all $N \gtrsim 2$. Finallly, For $I < I_c$, the system is ohmic with:

$$\langle V(I)\rangle = Z_0 I \tag{36}$$

As discussed in Section II, application of a small additional ac voltage produces features on the current-voltage characteristics for the currents that produce Bloch oscillation with frequencies commensurate with the frequency of the applied ac field $\omega_B = 2\pi\zeta I/2e = (m/n)\omega$. At these currents the system becomes insulating with respect to current increments, the largest such feature appears at $m = n = 1$ that allows a direct measurement of the Josephson element periodicity.

For smaller $E_J/E_C \sim 1$ the quasiclassical formulas for the transition amplitudes do not work and one has to perform the numerical diagonalization of the quantum system in order to find its actual spectrum. As $E_J/E_C \to 1$ the higher energy band approaches the low energy band and the dispersion of the latter deviates from the simple cosine form shown in Figure 3. These deviations, lead to higher harmonics in the dispersion: $\epsilon(q) = 2w\cos 2\pi\zeta q + 2w'\cos 4\pi\zeta q$ and change the equations (34,35). Our numerical diagonalization of a single rhombus shows, however, that even at relatively small $E_J/E_C \sim 1$ the second harmonics w' does not exceed $0.15w$, so its additional contribution to the voltage current characteristic ($\propto w'^2$) can always be neglected. Thus, in the whole range of $E_J/E_C > 1$ the voltage current characteristic is given by Eqs. (34,35) where the effective value of transition amplitude t can be found from the band width $W = E_1 - E_0 = 4w$ plotted in Fig. 3. For comparison we show the variation of the lower band width for a single junction in Fig. 4
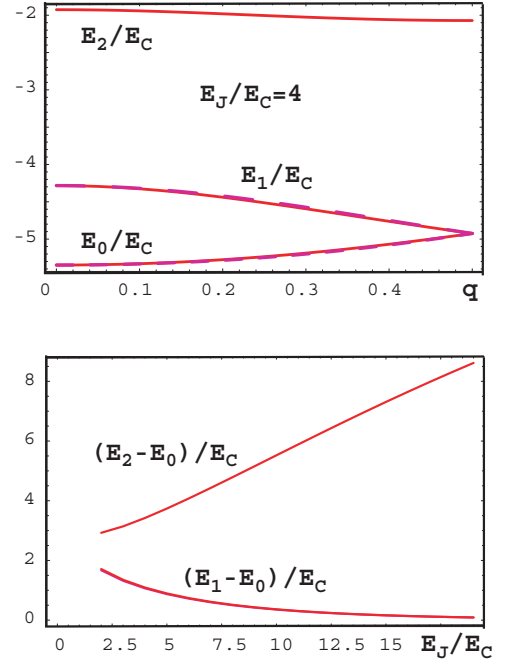

FIG. 3: Spectrum of a single rhombus biased by magnetic flux $\Phi = \Phi_0/2$. The upper pane shows the bands of the rhombus characterized by Josephson eneergy $E_J/E_C = 4$ as a function of bias charge, q. The two lower levels are fitted by the first two harmonics (dashed line), the coefficient w' of the second harmonics is $w' = 0.1w$. One observes period doubling of the first two states that reflects the symmetries of the rhombus frustrated by a half flux quantum. The second excited level is doubly degenerate that makes its period doubling difficult to observe. Physically, these states correspond to an excitation localized on the upper or lower arms of the rhombus. The lower pane shows the dependence of the gaps for $q = 0$ as a function of E_J/E_C. Because higher order harmonics are very small for all $E_J/E_C > 1$, the gap $E_1 - E_0$ coincides with $4w$ where w is the tunneling amplitude between the two classical ground states.

V. PHYSICAL IMPLEMENTATIONS

Generally, the effects described in the previous sections can be observed if the environment does not affect much the quantum fluctuations of individual elements and the resulting quasiclassical equations of motion. These physical requirements translate into different conditions on the impedance of the environment at different frequencies.

We begin with the quantum dynamics of the individual elements. The effect of the leads impedance on it can be taken into account by adding the appropriate current term to the phase equation of motion before projecting on a low energy band and requiring that their effect on the phase dynamics is small at the relevant frequencies. For instance, for a single junction

$$\frac{I}{2e} = E'_J(\phi) + \frac{1}{4E_c}\frac{d^2\phi}{dt^2} + \left[\frac{Z_Q}{Z_\omega}\right]\frac{d\phi}{dt}$$
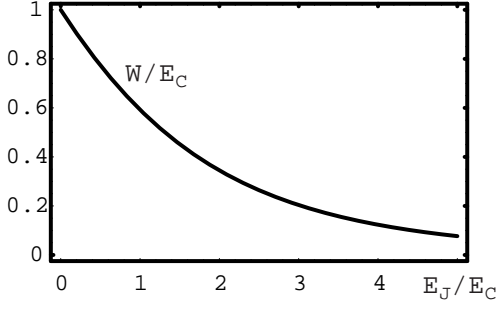
FIG. 4: Band width $W = 4w$ of a single Josephson junction

The characteristic frequency of the quantum fluctuations responsibe for the tunneling of a single element is Josephson plasma frequency, $\omega_J = \sqrt{8E_J E_c}$, so the first condition implies that

$$|Z(\omega_J)| \gg \sqrt{E_c/E_J} Z_Q \qquad (37)$$

For a typical $\omega_J/2\pi \sim 10 GHz$, the impedance of a simple superconducting lead of the length $\sim 1cm$ is smaller than Z_Q and the condition (37) is not satisfied. The situation is changed if the Josephson elements are decoupled from the leads by a large resistance or by a chain of $M \gg 1$ large junctions with $\sqrt{\widetilde{E}_J/\widetilde{E}_c} \gg 1$ that has no quantum tunneling transitions of their own (the amplitude of such transitions is $\propto \exp(-\sqrt{8\widetilde{E}_J/\widetilde{E}_c}$). Assuming that elements of this chain have no direct capacitive coupling to the ground $(M^2 C_0 \ll C)$, the chain has an impedance $Z = \sqrt{8\widetilde{E}_c/\widetilde{E}_J} M Z_Q$ at the relevant frequencies, so a realistic chain with $M \sim 50$ junctions and $\sqrt{8\widetilde{E}_J/\widetilde{E}_c} \sim 10$ provides the contribution to the impedance needed to satisfy (37).

Similar decoupling from the leads of the individual elements can be achieved by a sufficiently long chain of similar Josephson elements, e.g. rhombi. Consider a long $(N \gg 1)$ chain of similar elements connected to the leads characterized by a large but finite capacitance $C_g \gg C$. For a short chain the tunneling of a single element changes the phase on the leads resulting in a huge action of the tunneling process. However, in a long chain of junctions, a tunneling of individual rhombus may be compensated by a simultaneous change of the phases $\delta\phi/N$ of the remaining rhombi, and subsequent relaxation of $\delta\phi$ from its initial value π towards the equilibrium value which is zero. For $N \gg 1$, this later process can be treated within the Gaussian approximation, with the Lagrangian (in imaginary time):

$$L = \frac{1}{16E_g}\dot{\phi}^2 + \frac{1}{2N}E_J\phi^2 \qquad (38)$$

where $E_g = e^2/(2C_g)$. So the total action involved in the relaxation is: $S = \frac{\pi^2}{4\sqrt{2}} \left(\frac{E_J}{NE_g}\right)^{1/2}$ If this action S is less

than unity this relaxation has strictly no effect on the tunneling amplitude of the individual rhombus.

We now turn to the constraints imposed by the quasiclassical equations of motion. The solution of these equations shows oscillation at the Bloch frequency that is $\omega_B = 2\pi\zeta I/(2e)$ for large currents and approaches zero near the I_c. Thus, for a single Josephson element the quasiclassical equations of motion are valid if $\mathrm{Re}(R_Q/Z(\omega_B)) \ll 1$. A realistic energy band for a Josephson element, $W \sim 0.3K$ and $Z/Z_Q \sim 100$ correspond to Bloch frequency $\omega_B/2\pi \sim 0.1 GHz$. In this frequency range a typical lead gives a capacitive contribution to the dynamics. The condition that it does not affect significantly the equations of motion implies that the lead capacitance $C \lesssim 10 fF$. As discussed in Section (III) the individual elements in a short chain oscillate in antiphase decreasing the effective coupling to the leads by a factor $\sqrt{R/Z}$ where R is the intrinsic resistance of the contact. This factor can easily reach 10^2 at sufficiently low temperatures making much less restrictive the condition on the lead capacitance.

Large but finite impedance of the environment $\mathrm{Re}(R_Q/Z(\omega_B)) \lesssim 1$ modifes the observed current-voltage characteristics qualitatively, specially in the limit of very small driving current. When I vanishes, and with infinite external impedance, the wave function of the phase variable is completely extended, with the form of a Bloch state, and the pseudo-charge q is a good quantum number. As discussed at the end of Sec. II, the system behaves as a capacitor. But when the external impedance is finite, charge fluctuations appear, which in the dual description means that quantum phase fluctuations are no longer unbounded. To be specific, consider a realistic example of N rhombi chain (or two ordinary junctions) attached to the leads with $Z(\omega) = Z_0$ in a broad but finite frequency interval $\omega_{\min} < \omega < \omega_{\max}$ and decreases as $Z(\omega) = Z_0(\omega_{\max}/\omega)$ for $\omega > \omega_{\max}$, $Z(\omega) = Z_0(\omega/\omega_{\min})$ for $\omega < \omega_{\min}$. Such $Z(\omega)$ is realized in a long chain of M Josephson junctions between islands with a finite capacitive coupling to the ground C_0: $\omega_{\max} = \omega_J$ and $\omega_{\min} = (\sqrt{C/C_0}/M)\omega_J$. The effective action describing the phase dynamics across the chain has contributions from the tunneling of individual rhombi and from impedance of the chain

$$L_{tot} = \frac{1}{2}\left[\frac{\omega^2}{8\pi^2\zeta^2 Nw} + \frac{i\omega Z_Q}{Z(\omega)}\right]\phi^2$$

Here the first term describes the effect of the tunneling of the Josephson element between its quasiclassical minima which we approximate by a single tunneling amplitude w resulting in a spectrum $\epsilon(q) = -2w\cos 2\pi\zeta q$ that in a Gaussian approximation becomes $\epsilon(q) = 4\pi^2\zeta^2 wq^2$. This approximation is justified by the fact that, as we show below, the main effect of the phase fluctuations comes from the broad frequency range where the action is dominated by the second term while the first serves only as a cutoff of the logarithmical divergence. Its precise form is therefore largely irrelevant.

This action leads to a large but finite phase fluctuations

$$\langle \phi^2 \rangle = i \int \frac{d\omega}{2\pi} \frac{1}{\frac{\omega^2}{8\pi^2\zeta^2 Nw} + \frac{i\omega Z_Q}{Z(\omega)}} \approx \frac{Z_0}{R_Q} \ln \frac{\min(\omega_{\max}, \omega'_{\max})}{\omega_{\min}}$$

where $\omega'_{\max} = 8\pi^2\zeta^2 Nw(Z_Q/Z_0)$. These fluctuations are only logarithmically large, so they result in a finite renormalization of the Josephson energy of the rhombi chain and the corresponding critical current. In the absence of such renormalization the Josephson energy of a finite chain of elements can be approximated by the leading harmonics $E(\phi) = -E_0 \cos(\phi/\zeta)$ with $E_0 \sim E_J$ for $N \sim 1$ and $E_J \gtrsim E_c$. Renormalization by fluctuations replaces E_0 by

$$E_R = \exp(-\frac{1}{2}\langle \phi^2 \rangle)E_0 = \left[\frac{\min(\omega_{\max}, \omega'_{\max})}{\omega_{\min}} \right]^{-\frac{Z_0}{2R_Q}} E_0$$

In the limit of $\omega_{\min} \to 0$ or $Z_0 \to \infty$ the phase fluctuations renormalize Josephson energy to zero. But for realistic parameters this suppression of Josephson energy is finite which thus results in a small but non-zero value of the critical current. In this situation the current-voltage characterictics sketched in Fig. 1 is modified for very small values of currents and voltages: instead of insulating regime at very low currents and voltages one should observe a very small supercurrent ($E_R/2e$) followed by a small voltage step as shown in Fig. 2 by a dashed line. As is clear from the above discussion the value of the resulting critical current is controlled by the phase fluctuations at low $\omega \ll \omega_{\max}$; these frequencies are much smaller than the typical internal frequencies of a chain of Josephson elements which can be thus lumped together into an effective object characterized by the bare Josephson energy $E(\phi)$ and transition amplitude between its minima w. We thus expect the same qualitative behavior for a small chain of Josephson elements as for a single element at low currents.

VI. CONCLUSION

The main results of the present work are the expressions (34), (35) for the I-V curves of a chain of N identical basic Josephson circuits. They are derived within the assumption that the Josephson coupling is much larger than the charging energy, but in fact, the numerical calculations show that they remain very accurate even if $E_J \approx E_C$. These equations predict a maximum dc voltage when $I = I_c$ and $V(I) \propto 1/I$ for $I \gg I_c$. The anomalous V versus I dependence exhibited by these equations is a signature of underdamped quantum phase dynamics. It occurs only if the impedance of the external circuitry is sufficiently large both at the frequency of Bloch oscillations and at the Josephson frequency of the individual elements. The precise conditions are given in Section V. Observation of this dependence and the measurement of the maximal voltage would provide the proof of the quantum dynamics and the measurements of the tunneling amplitude which is the most important characteristics of these systems. It would also provide a crucial test of the quality of decoupling to the environment.

As a deeply quantum mechanical system, the chain of Josephson devices is very sensitive to an additional ac driving. It exhibits resonances when the driving frequency is commensurate with the frequency $\omega_B = 2\pi\zeta I/2e$ of the Bloch oscillations. This would provide additional ways to characterize the quantum dynamics of these circuits and confirm the period doubling of the rhombi frustrated by exactly half flux quantum.

Acknowledgments

APPENDIX A: QUANTUM-MECHANICAL CALCULATION OF THE DC VOLTAGE

In the large current regime where $I \geq I_{\max}$, the energy drop $\Delta_B = hI/2e$ induced by the driving current when ϕ increases by 2π becomes comparable to or larger than the bandwidth W. In this regime, the semi-classical approach is no longer reliable. But as long as Δ_B remains small compared to the typical gap Δ between nearby bands, we may still construct the system wave functions in the presence of the driving field from Wannier orbitals belonging to a single band. In such quantum-mechanical approach, dissipation is described as the result of coupling the single degree of freedom (ϕ, q) to a continuum of oscillator modes (q_α, p_α). The corresponding Hamiltonian has the form:

$$H_n = \epsilon_n \left(q - \sum_\alpha g_\alpha q_\alpha \right) - \frac{\hbar I}{2e}\phi + \sum_\alpha \frac{\hbar\omega_\alpha}{2}(q_\alpha^2 + p_\alpha^2) \quad \text{(A1)}$$

where we have chosen the following commutation relations:

$$[\phi, q] = i, \quad [q_\alpha, p_\beta] = i\delta_{\alpha\beta} \quad \text{(A2)}$$

and all other commutators between these operators vanish. The form of (A1) is plausible on the physical ground because when the superconducting islands are coupled to macroscopic leads, the charge q undergoes quantum fluctuations, so that it has to be replaced by a "dressed charge" $q - \sum_\alpha g_\alpha q_\alpha$ in the effective Hamiltonian. A more explicit justification is that the corresponding semiclassical equations of motion have the same form as Eqs. (4), (5) which simply mean that the effective current going through the superconducting circuit is the bias current minus the current going through the external impedence. The semi-classical equations deduced from (A1)

read:

$$\frac{d\phi}{dt} = \frac{1}{\hbar}\frac{d\epsilon_n}{dq}\left(q - \sum_\alpha g_\alpha q_\alpha\right)$$

$$\frac{dq}{dt} = \frac{I}{2e}$$

$$\frac{dq_\alpha}{dt} = \omega_\alpha p_\alpha$$

$$\frac{dp_\alpha}{dt} = -\omega_\alpha q_\alpha + \frac{g_\alpha}{\hbar}\frac{d\epsilon_n}{dq}\left(q - \sum_\alpha g_\alpha q_\alpha\right)$$

It is then natural to introduce $q' = q - \sum_\alpha g_\alpha q_\alpha$, so that:

$$\frac{d\phi}{dt} = \frac{1}{\hbar}\frac{d\epsilon_n}{dq}(q') \tag{A3}$$

$$\frac{dq'}{dt} = \frac{I}{2e} - \sum_\alpha g_\alpha \frac{dq_\alpha}{dt} \tag{A4}$$

To show that (A4) has the same form as (5), we notice that the driving term for the bath oscillators is directly proportional to $d\phi/dt$. Specifically:

$$\frac{dq_\alpha}{dt} + \omega_\alpha^2 q_\alpha = \omega_\alpha g_\alpha \frac{d\phi}{dt} \tag{A5}$$

Going to Fourier space, we see that after averaging over initial conditions for the bath oscillators, Eq. (A4) takes the form:

$$-i\omega\tilde{q}'(\omega) = \frac{I}{2e}2\pi\delta(\omega) - i\sum_\alpha g_\alpha^2 \frac{\omega\omega_\alpha}{\omega^2 - \omega_\alpha^2}\left(-i\omega\tilde\phi(\omega)\right) \tag{A6}$$

This is exactly the frequency space version of Eq. (5), where as usual, the dissipation is related to the spectral density of the bath by:

$$i\sum_\alpha g_\alpha^2 \frac{\omega\omega_\alpha}{\omega^2 - \omega_\alpha^2} = \frac{Z_Q}{Z(\omega)} \tag{A7}$$

Here we emphasize that Z is typically frequency dependent, in which case the term $(Z_Q/Z)(d\phi/dt)$ in Eq. (5) becomes a convolution with Z_Q/Z replaced by a non-local kernel in time.

Now we turn to the solution of the quantum problem (A1) in the large driving current regime. Let us first consider the Josephson array without dissipation. It is straightforward to express its eigenstates in the q representation[41] because the Schrödinger equation reads then:

$$E\psi(q) = \epsilon_n(q)\psi(q) - i\frac{\hbar I}{2e}\frac{d\psi}{dq}(q) \tag{A8}$$

so that:

$$\psi(q) = \psi(0)\exp\left(-i\frac{2e}{\hbar I}\int_0^q (\epsilon_n(q') - E)dq'\right) \tag{A9}$$

The energy spectrum is determined via the boundary condition $\psi(q+1) = \psi(q)$ so that:

$$E_\nu = \int_{-1/2}^{1/2}\epsilon_n(q')dq' + \Delta_{WS}\nu, \quad \nu \text{ integer} \tag{A10}$$

This yields a Wannier-Stark ladder of spacially localized states, with a constant level spacing equal to Δ_B. Note that increasing ν by one unit multiplies the wave-function $\psi(q)$ by $\exp(i2\pi q)$. In the phase representation, this is equivalent to a translation by -2π. Of course, in the absence of dissipation, these levels have infinite life-time, and therefore, no dc voltage is generated.

Let us now consider the limit of a weak coupling to the dissipative bath. This means that the decay rate Γ of the Wannier-Stark levels is much smaller than the level spacing Δ_B. Assuming that transitions take place mostly between two adjacent levels, we get an average voltage:

$$\langle V \rangle = \frac{\hbar}{2e}\langle\frac{d\phi}{dt}\rangle = \frac{h\Gamma}{2e} \tag{A11}$$

The rate Γ is estimated via Fermi's golden rule which we prefer to use in the correlation function formulation :

$$\Gamma_{\nu \to \nu'} = \frac{1}{\hbar^2}|\langle\nu|\frac{d\epsilon_n}{dq}|\nu'\rangle|^2 \tilde{C}_{AA}((\nu-\nu')\omega_B) \tag{A12}$$

where $\omega_B = \Delta_B/\hbar = 2\pi I/(2e)$. In this expression, $\tilde{C}_{AA}$ is the Fourier transform of the correlation function, $C_{AA}(t - t') = \langle A(t)A(t')\rangle$ of the Heisenberg operators $A(t) = \sum_\alpha g_\alpha q_\alpha(t)$, taken in the equilibrium state of the dissipative bath.

We evaluate now the matrix element of the velocity operator $d\epsilon_n/dq$ between Wannier-Stark states:

$$\langle\nu|\frac{d\epsilon_n}{dq}|\nu'\rangle = \int_{-1/2}^{1/2}\frac{d\epsilon_n}{dq}(q)\exp(i2\pi(\nu'-\nu)q)\,dq \tag{A13}$$

As we have seen, in most physically interesting situations, we can approximate the periodic function $\epsilon(q)$ by a single harmonic $2w\cos(2\pi q)$. In this case:

$$\langle\nu|\frac{d\epsilon_n}{dq}|\nu'\rangle = 2\pi w \delta_{|\nu'-\nu|,1} \tag{A14}$$

In the zero temperature limit, the bath correlation function is:

$$\tilde{C}_{AA}(\omega) = \sum_\alpha \pi g_\alpha^2 \delta(\omega - \omega_\alpha) = \frac{2}{\omega}\text{Re}\left(\frac{Z}{Z_\omega}\right)\theta(\omega) \tag{A15}$$

where $\theta(\omega)$ is the Heaviside step function. Putting all these elements together gives:

$$\langle V(I \gg I_c)\rangle = \frac{(2\pi w)^2}{2e^2 Z_\omega I} \tag{A16}$$

where Z_ω denotes the external impedance taken for $\omega = \omega_B$, and this result is in perfect agreement with the large current limit of the semi-classical treatment, shown as Eq. (35).

When the band structure is replaced by $2w\cos(2\pi\zeta q)$ as in the case of a rhombus (for which $\zeta = 1/2$), we have now:

$$\langle V \rangle = \zeta\frac{h\Gamma}{2e} \tag{A17}$$

The frequency of Bloch oscillations becomes $\omega_{\mathrm{B}}(\zeta) = \zeta\omega_{\mathrm{B}}(\zeta = 1)$, and the matrix element is multiplied by ζ, so that the voltage is multiplied by ζ^2. Again, this is compatible with the semi-classical result (35).

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